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1996 J. Phys. A: Math. Gen. 29 8163

(http://iopscience.iop.org/0305-4470/29/24/037)

J. Phys. A: Math. Gen. **29** (1996) 8163–8167. Printed in the UK

Integrability of the $N = 3$ super KdV equation

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Received 13 August 1996

Abstract. We obtain the Lax pair for the $N = 3$ supersymmetric and $SO(3)$ symmetric extensions of KdV equation by using the prolongation structure technique. Hence its integrability is proved.

The $N = 1$ and $N = 2$ integrable supersymmetric (fermionic) extensions of the KdV equation have been widely studied [1–4] because they are related to the superconformal (super Virasoro) algebras, which play a central role in superstring theory, 2D critical phenomena and supergravities, and so on. Recently Bellucci *et al* investigated in detail the most general $SO(3)$ symmetric $N = 3$ superfield extension of the KdV equation [5]. They found that only a candidate exists for a non-trivial, higher-order conservation law and conjectured it to be completely integrable. In this paper we prove this conjecture by using the prolongation structure technique [6–8].

The $N = 3$ super KdV equation [5] reads

$$J_t = -J_{xxx} + 3(JD^3J)_x + \tfrac{3}{2}J(D^iJD^iJ)_x \tag{1}$$

where the superfield $J(x,t,\theta) = \psi(x,t) + \theta^i v_i(x,t) + \theta^{3-i}\varphi_i(x,t) + \theta^3 u(x,t)$, θ^i ($i = 1,2,3$) are anticommuting variables, $\theta^3 = \frac{1}{6}\epsilon_{kji}\theta^i\theta^j\theta^k$, $\theta^{3-i} = \frac{1}{2}\epsilon_{kji}\theta^j\theta^k$, $D^i = \partial_{\theta^i} - \theta^i\partial$, $D^3 = \frac{1}{6}\epsilon_{ijk}D^iD^jD^k$, u and v_i are the ordinary bosonic (modified) KdV fields, and ψ and φ_i are the fermionic fields. After suitable reduction to the $N = 2$ case, equation (1) becomes the $N = 2$ supersymmetric KdV equation with the parameter $a = 1$ [4], which is shown to be integrable [9, 8]. According to the prolongation structure technique [6], we first rewrite equation (1) equivalently as the following 2-forms:

$$\begin{aligned}
&\alpha_1 = \mathrm{d}t\wedge\mathrm{d}u + \mathrm{d}x\wedge\mathrm{d}t\,u_x \qquad \alpha_2 = \mathrm{d}t\wedge\mathrm{d}u_x + \mathrm{d}x\wedge\mathrm{d}t\,u_{xx}\\
&\alpha_3 = \mathrm{d}x\wedge\mathrm{d}u - \mathrm{d}t\wedge\mathrm{d}u_{xx} - \mathrm{d}t\wedge\psi_{xxx}3\psi - \mathrm{d}t\wedge\mathrm{d}v_{ixx}3v_i - \mathrm{d}x\wedge\mathrm{d}t\,A\\
&\alpha_{3+i} = \mathrm{d}t\wedge\mathrm{d}v_i + \mathrm{d}x\wedge\mathrm{d}t\,v_{ix} \qquad \alpha_{6+i} = \mathrm{d}t\wedge\mathrm{d}v_{ix} + \mathrm{d}x\wedge\mathrm{d}t\,v_{ixx}\\
&\alpha_{9+i} = \mathrm{d}x\wedge\mathrm{d}v_i - \mathrm{d}t\wedge\mathrm{d}v_{ixx} - \mathrm{d}x\wedge\mathrm{d}t\,B_i \qquad \beta_1 = \mathrm{d}t\wedge\mathrm{d}\psi + \mathrm{d}x\wedge\mathrm{d}t\,\psi_x\\
&\beta_2 = \mathrm{d}t\wedge\mathrm{d}\psi_x + \mathrm{d}x\wedge\mathrm{d}t\,\psi_{xx} \qquad \beta_3 = \mathrm{d}x\wedge\mathrm{d}\psi_{xx} + \mathrm{d}x\wedge\mathrm{d}t\,\psi_{xxx}\\
&\beta_4 = \mathrm{d}x\wedge\mathrm{d}\psi - \mathrm{d}t\wedge\mathrm{d}\psi_{xx} - \mathrm{d}x\wedge\mathrm{d}t\,C \qquad \beta_{4+i} = \mathrm{d}t\wedge\mathrm{d}\varphi_i + \mathrm{d}x\wedge\mathrm{d}t\,\varphi_{ix}\\
&\beta_{7+i} = \mathrm{d}t\wedge\mathrm{d}\varphi_{ix} + \mathrm{d}x\wedge\mathrm{d}t\,\varphi_{ixx}\\
&\beta_{10+i} = \mathrm{d}x\wedge\mathrm{d}\varphi_i - \mathrm{d}t\wedge\mathrm{d}\varphi_{ixx} - \mathrm{d}t\wedge\mathrm{d}v_{ixx}3\psi - \mathrm{d}x\wedge\mathrm{d}t\,D_i
\end{aligned} \tag{2}$$

where

$$A = 3\psi_x\psi_{xxx} + 6uu_x + 3\varphi_i\varphi_{ixx} - 3v_{ix}v_{ixx} - 3\psi(2\varphi_i v_{ix} - v_i\varphi_{ix} - 3u\psi_x)_x + 3(uv_iv_i)_x$$
$$+\tfrac{3}{2}\epsilon_{ijk}v_i(\varphi_j\varphi_k)_x - 6v_i(\varphi_i\psi_x)_x + 3\varphi_i(v_i\psi_x)_x - 3\epsilon_{ijk}\varphi_i(v_j\varphi_k)_x$$
$$B_i = 3(uv_i + \psi\varphi_{ix})_x + 3\psi(v_i\psi_x - \epsilon_{ijk}v_j\varphi_k)_x + 3v_iv_jv_{jx} \tag{3}$$
$$C = 3(u\psi)_x + 3\psi v_iv_{ix}$$
$$D_i = -3\psi_x v_{ixx} + 3(u\varphi_i + \epsilon_{ijk}v_j\varphi_{kx})_x + 3\psi(uv_i + \tfrac{1}{2}\epsilon_{ijk}\varphi_j\varphi_k - 2\varphi_i\psi_x + \epsilon_{ijk}v_jv_{kx})_x$$
$$+3\varphi_iv_jv_{jx} + 3\epsilon_{ijk}\psi_xv_jv_{kx} + 3\epsilon_{ijk}\epsilon_{lmk}v_j(\varphi_lv_m)_x.$$

It is easy to prove that these even 2-forms α_m and odd 2-forms β_n form a closed ideal I, i.e. $\mathrm{d}I \subset I$. Then, we prolong I to $\widetilde{I}$ by adding the following set of even and odd 1-forms [7, 8]:

$$\omega_k^0 = \mathrm{d}y_k + \mathrm{d}x F_k^0 + \mathrm{d}t G_k^0 \qquad k = 1, 2, \ldots, n_0$$
$$\omega_l^1 = \mathrm{d}\zeta_l + \mathrm{d}x F_l^1 + \mathrm{d}t G_l^1 \qquad l = 1, 2, \ldots, n_1 \tag{4}$$

where y_k and ζ_l are the even and odd prolonged variables, respectively, F_k^0 and G_k^0 are even functions, and F_l^1 and G_l^1 are odd functions in terms of the variables $u, u_x, u_{xx}, \psi, \psi_x, \psi_{xx}, \psi_{xxx}, v_i, v_{ix}, v_{ixx}, \varphi_i, \varphi_{ix}, \varphi_{ixx}, y_k$ and ζ_l. Let $\widetilde{I}$ also be closed; then we have

$$F_{u_x} = F_{u_{xx}} = F_{\psi_x} = F_{\psi_{xx}} = F_{\psi_{xxx}} = F_{v_{ix}} = F_{v_{ixx}} = F_{\varphi_{ix}} = F_{\varphi_{ixx}} = 0$$
$$G_{u_{xx}} = -F_u \qquad G_{\psi_{xx}} = -F_\psi \qquad G_{\psi_{xxx}} = -3\psi F_u$$
$$G_{v_{ixx}} = -F_{v_i} - 3\psi F_{\varphi_i} - 3v_iF_u \qquad G_{\varphi_{ixx}} = -F_{\varphi_i} \tag{5}$$
$$u_xG_u + u_{xx}G_{u_x} + \psi_xG_\psi + \psi_{xx}G_{\psi_x} + \psi_{xxx}G_{\psi_{xx}} + v_{ix}G_{v_i} + v_{ixx}G_{v_{ix}}$$
$$+\varphi_{ix}G_{\varphi_i} + \varphi_{ixx}G_{\varphi_{ix}} - AF_u - B_iF_{v_i} - CF_\psi - D_iF_{\varphi_i} - [F, G] = 0$$

where $F = F_k^0\partial_{y_k} + F_l^1\partial_{\zeta_l}$ and $G = G_k^0\partial_{y_k} + G_l^1\partial_{\zeta_l}$.

For equations (5), we obtain the solutions

$$F = v_iX_i + uX_4 + \varphi_iX_{-i} + \psi X_{-4}$$
$$G = -H_iX_i - IX_4 + J_iX_{4+i} - KX_{-4} - L_iX_{-i} + M_iX_{-4-i} + PX_{-8}. \tag{6}$$

Here

$$H_i = v_{ixx} - 3uv_i - 3\psi\varphi_{ix} - 3v_i\psi\psi_x - v_iv_jv_j + 2\epsilon_{ijk}\psi v_j\varphi_k$$
$$I = u_{xx} - 3\psi\psi_{xxx} + 3v_iv_{ixx} - 3u^2 - 3\varphi_i\varphi_{ix} - 3uv_iv_i - 3\epsilon_{ijk}v_i\varphi_j\varphi_k - 9u\psi\psi_x$$
$$+3(v_i\varphi_i\psi)_x - 9v_{ix}\varphi_i\psi$$
$$J_i = \epsilon_{ijk}v_jv_{kx} + \varphi_{ix}\psi - \varphi_i\psi_x$$
$$K = \psi_{xx} - 3u\psi - \psi v_iv_i \tag{7}$$
$$L_i = \varphi_{ixx} + 3\psi v_{ixx} - 3u\varphi_i - 3\epsilon_{ijk}v_j\varphi_{kx} - 6\varphi_i\psi\psi_x - 3\epsilon_{ijk}\psi v_jv_{kx} - 2uv_i\psi$$
$$-\epsilon_{ijk}\varphi_j\varphi_k\psi - 3\varphi_iv_jv_j + 2v_iv_j\varphi_j$$
$$M_i = v_i\psi_x - v_{ix}\psi$$
$$P = u\psi_x - u_x\psi + v_i\varphi_{ix} - v_{ix}\varphi_i.$$

Table 1. Commutation relations of the even generators X_p $(p = 1, 2, \ldots, 7)$ and odd generators X_{-q} $(q = 1, 2, \ldots, 8)$.

	X_{-8}	X_{-7}	X_{-6}	X_{-5}	X_{-4}	X_{-3}	X_{-2}	X_{-1}	X_1	X_2	X_3	X_4	X_5	X_6	X_7
X_{-8}	0	$-X_7$	$-X_6$	$-X_5$	0	0	0	0	$-X_{-1}$	$-X_{-2}$	$-X_{-3}$	0	0	0	0
X_{-7}		0	0	0	0	0	X_1	$-X_2$	0	0	$-X_{-4}$	$-X_{-3}$	$-X_{-6}$	X_{-5}	0
X_{-6}			0	0	0	$-X_1$	0	X_3	0	$-X_{-4}$	0	$-X_{-2}$	X_{-7}	0	$-X_{-5}$
X_{-5}				0	0	X_2	$-X_3$	0	$-X_{-4}$	0	0	$-X_{-1}$	0	$-X_{-7}$	X_{-6}
X_{-4}					0	$-X_7$	$-X_6$	$-X_5$	X_{-5}	X_{-6}	X_{-7}	X_{-8}	0	0	0
X_{-3}						0	0	0	0	0	X_{-8}	0	$-X_{-2}$	X_{-1}	0
X_{-2}							0	0	0	X_{-8}	0	0	X_{-3}	0	$-X_{-1}$
X_{-1}								0	X_{-8}	0	0	0	0	$-X_{-3}$	X_{-2}
X_1									0	$-X_7$	X_6	0	0	$-X_3$	X_2
X_2										0	$-X_5$	0	X_3	0	$-X_1$
X_3											0	0	$-X_2$	X_1	0
X_4												0	0	0	0
X_5													0	$-X_7$	X_6
X_6														0	$-X_5$
X_7															0

The even generators X_p $(p = 1, 2, \ldots, 7)$ and odd generators X_{-q} $(q = 1, 2, \ldots, 8)$ form a non-trivial Lie superalgebra. Their commutation relations are listed in table 1.

The representation of this Lie superalgebra is

$$
\begin{aligned}
&X_1 = \zeta_4\partial_{\zeta_1} - \zeta_1\partial_{\zeta_4} \qquad X_2 = \zeta_4\partial_{\zeta_2} - \zeta_2\partial_{\zeta_4} \qquad X_3 = \zeta_4\partial_{\zeta_3} - \zeta_3\partial_{\zeta_4}\\
&X_4 = \partial_y \qquad X_5 = \zeta_2\partial_{\zeta_3} - \zeta_3\partial_{\zeta_2} \qquad X_6 = \zeta_3\partial_{\zeta_1} - \zeta_1\partial_{\zeta_3}\\
&X_7 = \zeta_1\partial_{\zeta_2} - \zeta_2\partial_{\zeta_1} \qquad X_{-1} = \partial_{\zeta_1} \qquad X_{-2} = \partial_{\zeta_2} \qquad X_{-3} = \partial_{\zeta_3}\\
&X_{-4} = -\zeta_1\zeta_2\partial_{\zeta_3} - \zeta_2\zeta_3\partial_{\zeta_1} - \zeta_3\zeta_1\partial_{\zeta_2} + y\partial_{\zeta_4}\\
&X_{-5} = \zeta_2\zeta_3\partial_{\zeta_4} + \zeta_3\zeta_4\partial_{\zeta_2} + \zeta_4\zeta_2\partial_{\zeta_3} + y\partial_{\zeta_1}\\
&X_{-6} = -\zeta_3\zeta_4\partial_{\zeta_1} - \zeta_4\zeta_1\partial_{\zeta_3} - \zeta_1\zeta_3\partial_{\zeta_4} + y\partial_{\zeta_2}\\
&X_{-7} = \zeta_1\zeta_2\partial_{\zeta_4} + \zeta_2\zeta_4\partial_{\zeta_1} + \zeta_4\zeta_1\partial_{\zeta_2} + y\partial_{\zeta_3} \qquad X_{-8} = -\partial_{\zeta_4}.
\end{aligned}
\tag{8}
$$

On a solution manifold of $\widetilde{I}$, we have $\widetilde{\omega_k^0} = \widetilde{\omega_l^1} = 0$, i.e. $y_{kx} = -F_k^0$, $y_{kt} = -G_k^0$, $\zeta_{lx} = -F_l^1$ and $\zeta_{lt} = -G_l^1$. Substituting equation (8) in (6), we have

$$
\begin{aligned}
&y_x = -u \qquad \zeta_{ix} = -v_i\zeta_4 - \varphi_i + \tfrac{1}{2}\epsilon_{ijk}\psi\zeta_j\zeta_k \qquad \zeta_{4x} = v_i\zeta_i - y\psi\\
&y_t = I \qquad \zeta_{it} = H_i\zeta_4 - \tfrac{1}{2}\epsilon_{ijk}K\zeta_j\zeta_k + L_i - \epsilon_{ijk}J_j\zeta_k - M_i y + \epsilon_{ijk}M_j\zeta_k\zeta_4\\
&\zeta_{4t} = -H_i\zeta_i + Ky - \tfrac{1}{2}\epsilon_{ijk}M_i\zeta_j\zeta_k + P.
\end{aligned}
\tag{9}
$$

We can easily verify that $y_{xt} = y_{tx}, \zeta_{lxt} = \zeta_{ltx}, (l = 1, 2, 3, 4)$ hold if u, v_i, ψ and φ_i satisfy (1). Taking $y = -\partial^{-1}u = lnT_0 - \lambda x$, $T_1 = \zeta_1 T_0$, $T_2 = \zeta_2 T_0$, $T_3 = \zeta_3 T_0$, $T_4 = \zeta_4 T_0$, $T_5 = \zeta_1\zeta_2 T_0$, $T_6 = \zeta_1\zeta_3 T_0$, $T_7 = \zeta_2\zeta_3 T_0$, $T_8 = \zeta_1\zeta_4 T_0$, $T_9 = \zeta_2\zeta_4 T_0$, $T_{10} = \zeta_3\zeta_4 T_0$, $T_{11} = \zeta_1\zeta_2\zeta_3 T_0$, $T_{12} = \zeta_1\zeta_2\zeta_4 T_0$, $T_{13} = \zeta_1\zeta_3\zeta_4 T_0$, $T_{14} = \zeta_2\zeta_3\zeta_4 T_0$, $T_{15} = \zeta_1\zeta_2\zeta_3\zeta_4 T_0$ and using (9), we obtain the Lax representation for (1):

$$
L\mathbf{T} = \lambda\mathbf{T} \qquad \mathbf{T}_t = M\mathbf{T} \tag{10}
$$

where λ is an arbitrary constant, $\mathbf{T} = (T_0, T_1, \ldots, T_{15})^\top$, $L = (\partial + u) \cdot \mathbb{1} + \widetilde{L}$, $M = I \cdot \mathbb{1} + \widetilde{M}$, $\mathbb{1}$ is a 16×16 identity matrix and

$$\widetilde{L} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\varphi_1 & 0 & 0 & 0 & v_1 & 0 & 0 & -\psi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\varphi_2 & 0 & 0 & 0 & v_2 & 0 & \psi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\varphi_3 & 0 & 0 & 0 & v_3 & -\psi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\mu & -v_1 & -v_2 & -v_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\varphi_2 & \varphi_1 & 0 & 0 & 0 & 0 & 0 & v_2 & -v_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\varphi_3 & 0 & \varphi_1 & 0 & 0 & 0 & 0 & v_3 & 0 & -v_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\varphi_3 & \varphi_2 & 0 & 0 & 0 & 0 & 0 & v_3 & -v_2 & 0 & 0 & 0 & 0 & 0 \\
0 & \mu & 0 & 0 & \varphi_1 & -v_2 & -v_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\psi & 0 \\
0 & 0 & \mu & 0 & \varphi_2 & v_1 & 0 & -v_3 & 0 & 0 & 0 & 0 & 0 & \psi & 0 & 0 \\
0 & 0 & 0 & \mu & \varphi_3 & 0 & v_1 & v_2 & 0 & 0 & 0 & 0 & -\psi & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \varphi_3 & -\varphi_2 & \varphi_1 & 0 & 0 & 0 & 0 & v_3 & -v_2 & v_1 & 0 \\
0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & -\varphi_2 & \varphi_1 & 0 & -v_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & -\varphi_3 & 0 & \varphi_1 & v_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & -\varphi_3 & \varphi_2 & -v_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & \varphi_3 & -\varphi_2 & \varphi_1 & 0
\end{bmatrix}$$

$$\widetilde{M} =$$

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
R_1 & 0 & J_3 & -J_2 & H_1 & 0 & 0 & -K & 0 & -M_3 & M_2 & 0 & 0 & 0 & 0 & 0 \\
R_2 & -J_3 & 0 & J_1 & H_2 & 0 & K & 0 & M_3 & 0 & -M_1 & 0 & 0 & 0 & 0 & 0 \\
R_3 & J_2 & -J_1 & 0 & H_3 & -K & 0 & 0 & -M_2 & M_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
S & -H_1 & -H_2 & -H_3 & 0 & -M_3 & M_2 & -M_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -R_2 & R_1 & 0 & 0 & 0 & J_1 & J_2 & H_2 & -H_1 & 0 & 0 & 0 & M_1 & M_2 & 0 \\
0 & -R_3 & 0 & R_1 & 0 & -J_1 & 0 & J_3 & H_3 & 0 & -H_1 & 0 & -M_1 & 0 & M_3 & 0 \\
0 & 0 & -R_3 & R_2 & 0 & -J_2 & -J_3 & 0 & 0 & H_3 & -H_2 & 0 & w - M_2 & -M_3 & 0 & 0 \\
0 & -S & 0 & 0 & R_1 & -H_2 & -H_3 & 0 & 0 & J_3 & -J_2 & M_1 & 0 & 0 & -K & 0 \\
0 & 0 & -S & 0 & R_2 & H_1 & 0 & -H_3 & -J_3 & 0 & J_1 & M_2 & 0 & K & 0 & 0 \\
0 & 0 & 0 & -S & R_3 & 0 & H_1 & H_2 & J_2 & -J_1 & 0 & M_3 & -K & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & R_3 & -R_2 & R_1 & 0 & 0 & 0 & 0 & H_3 & -H_2 & H_1 & 0 \\
0 & 0 & 0 & 0 & 0 & S & 0 & 0 & -R_2 & R_1 & 0 & -H_3 & 0 & J_1 & J_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & S & 0 & -R_3 & 0 & R_1 & H_2 & -J_1 & 0 & J_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & S & 0 & -R_3 & R_2 & -H_1 & -J_2 & -J_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -S & R_3 & -R_2 & R_1 & 0
\end{bmatrix}$$

Here, $\mu = \psi \partial^{-1} u$, $R_i = L_i + M_i \partial^{-1} u$ and $S = P - K \partial^{-1} u$. From equation (10), we have $L_t = [M, L]$, which gives rise to equation (1). Therefore, the existence of the Lax operator L proves the integrability of (1).

Finally, we consider the bosonic core (i.e. all the fermionic fields ψ and φ_i vanish) of the $N = 3$ super KdV equation (1), which is

$$\begin{aligned}
u_t &= -u_{xxx} + 3(u^2 - v_i v_{ixx} + u v_i v_i)_x \\
v_{it} &= -v_{ixxx} + 3(u v_i)_x + 3 v_i v_j v_{jx}.
\end{aligned} \qquad (11)$$

The algebra generated by the prolongation structure of (11) is the bosonic part of the Lie superalgebra above. In this case, $X_5 = X_1$, $X_6 = X_2$ and $X_7 = X_3$. X_4 is central element commuting with the other generators and X_i form the $SO(3)$ algebra. The Lax pair associated with (11) is

$$L = \begin{bmatrix} \partial + u & 0 & 0 & 0 \\ 0 & \partial + u & -v_3 & v_2 \\ 0 & v_3 & \partial + u & -v_1 \\ 0 & -v_2 & v_1 & \partial + u \end{bmatrix}$$

$$M = \begin{bmatrix} U & 0 & 0 & 0 \\ 0 & U & -W_3 & W_2 \\ 0 & W_3 & U & -W_1 \\ 0 & -W_2 & W_1 & U \end{bmatrix}$$

where $U = u_{xx} + 3v_i v_{ixx} - 3u^2 - 3uv_i v_i$ and $W_i = v_{ixx} - \epsilon_{ijk} v_j v_{kx} - 3uv_i - v_i v_j v_j$.

Acknowledgments

This work is supported by the Sichuan Youth Science and Technology Foundation, the Excellent Youth Teacher Foundation of the Chinese National Educational Commission and the Foundation of Sichuan Educational Commission.

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